Session Types with Arithmetic Refinements

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What are Session Types?

- Implement message-passing concurrent programs
- Communication via typed bi-directional channels
- Communication protocol enforced by session types

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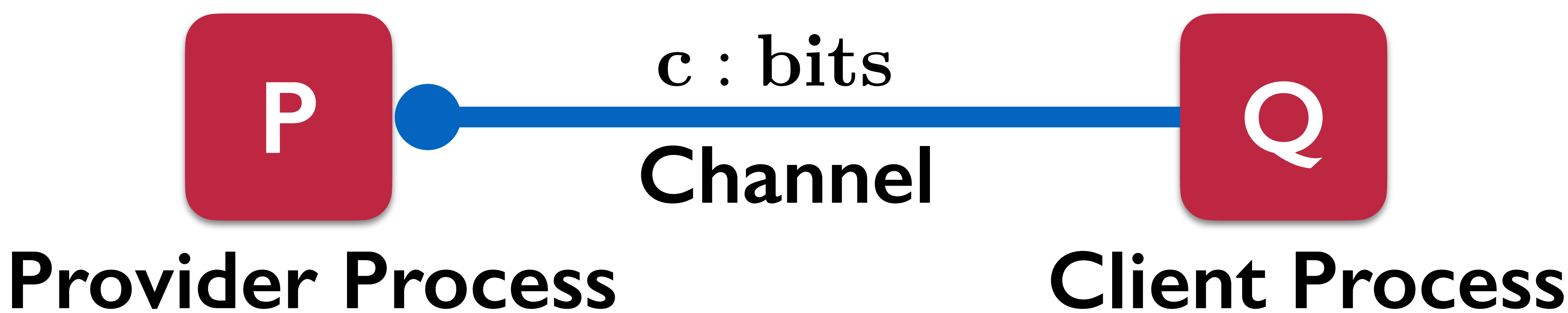

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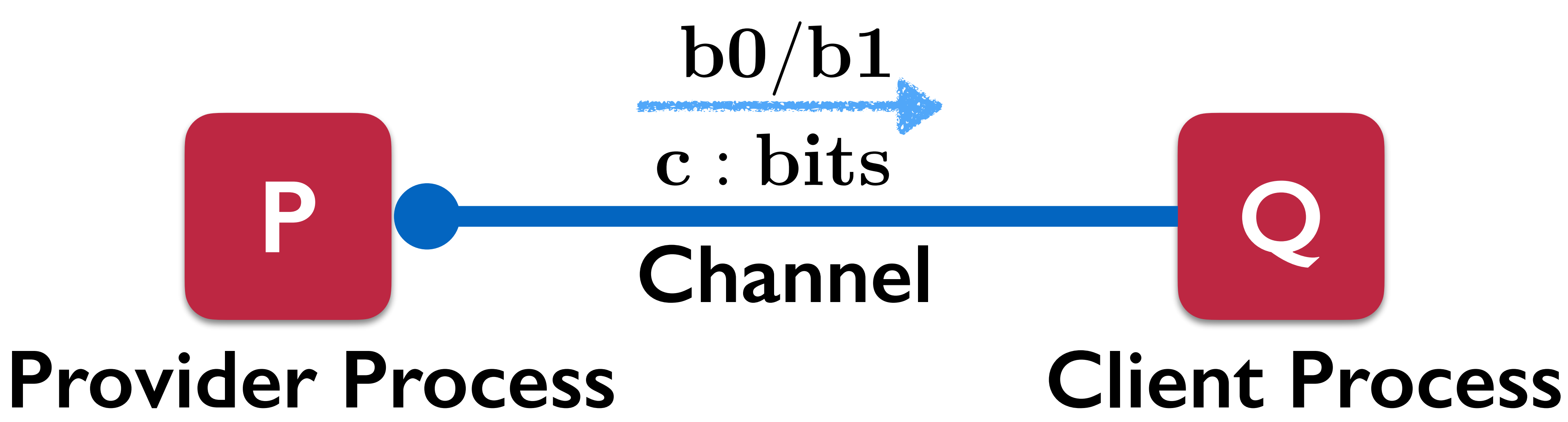
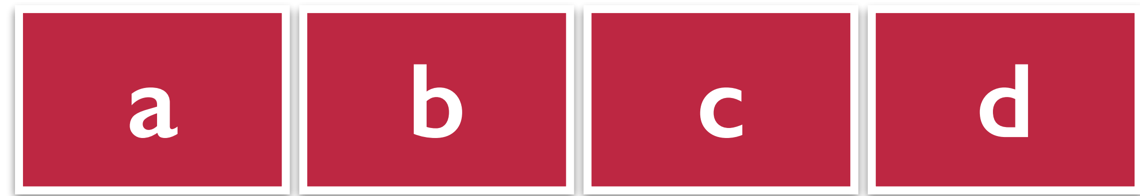


Example: Queues



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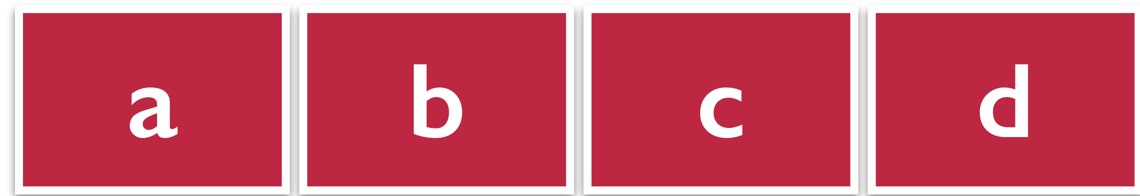

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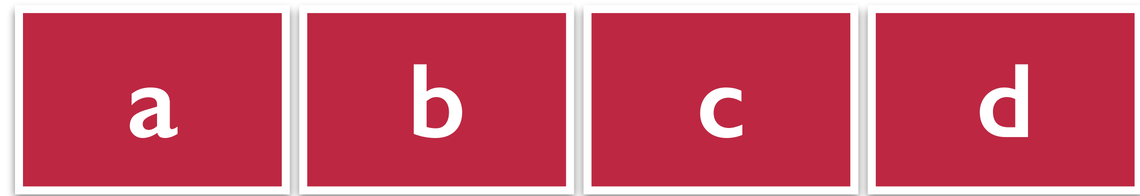

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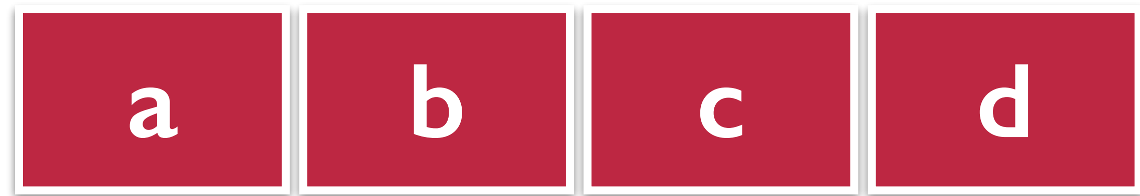

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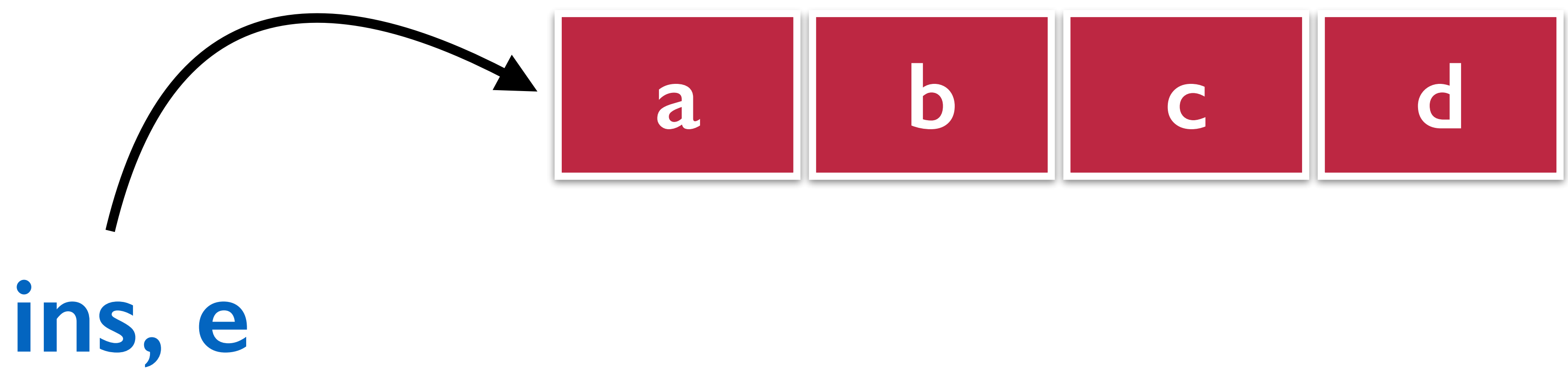

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a	b	c	d	e

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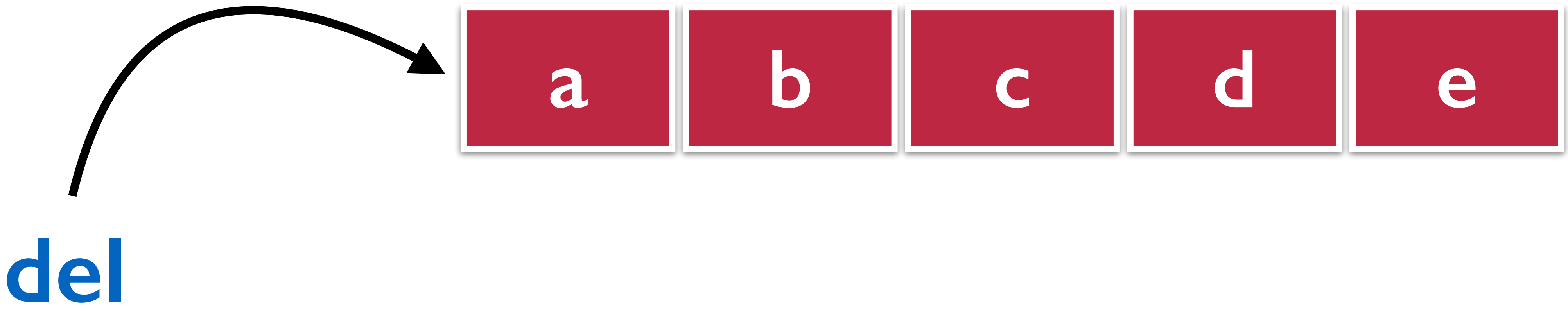

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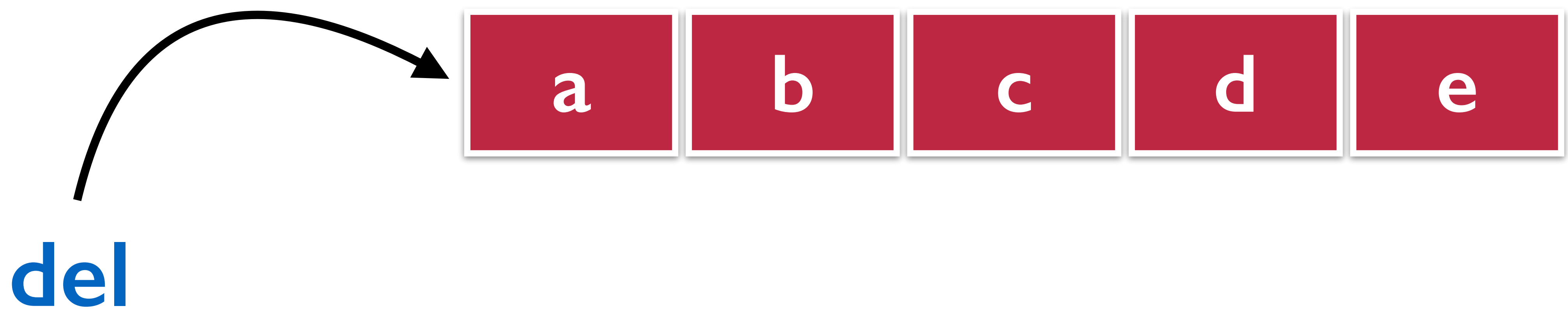

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send none if queue is empty

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terminate

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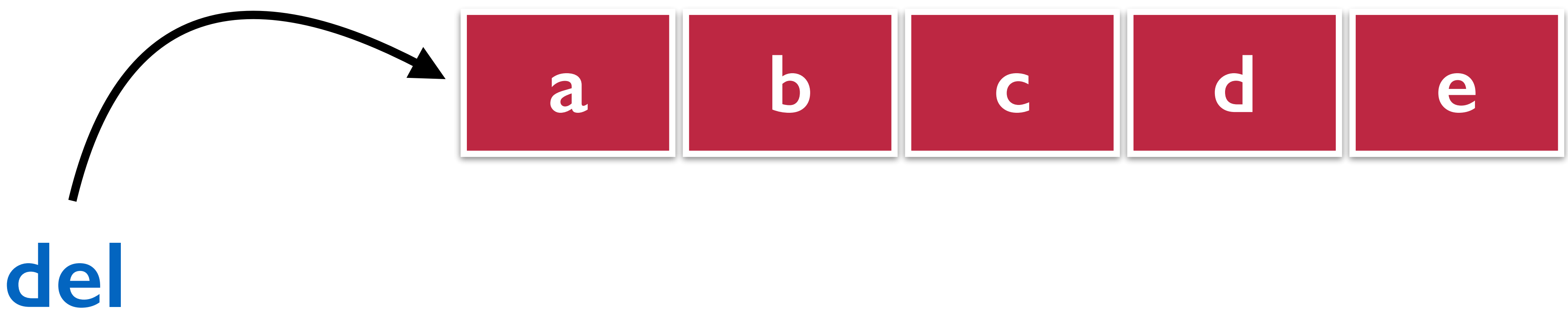

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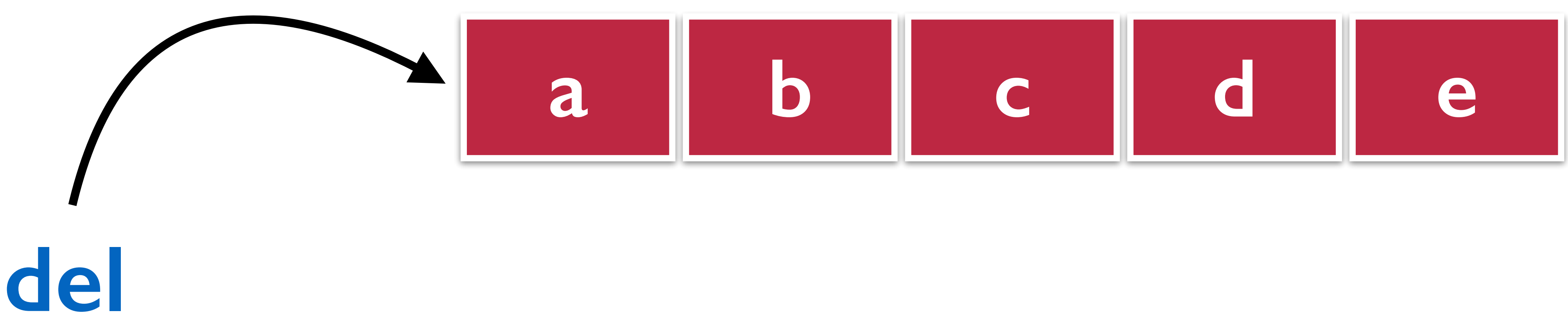

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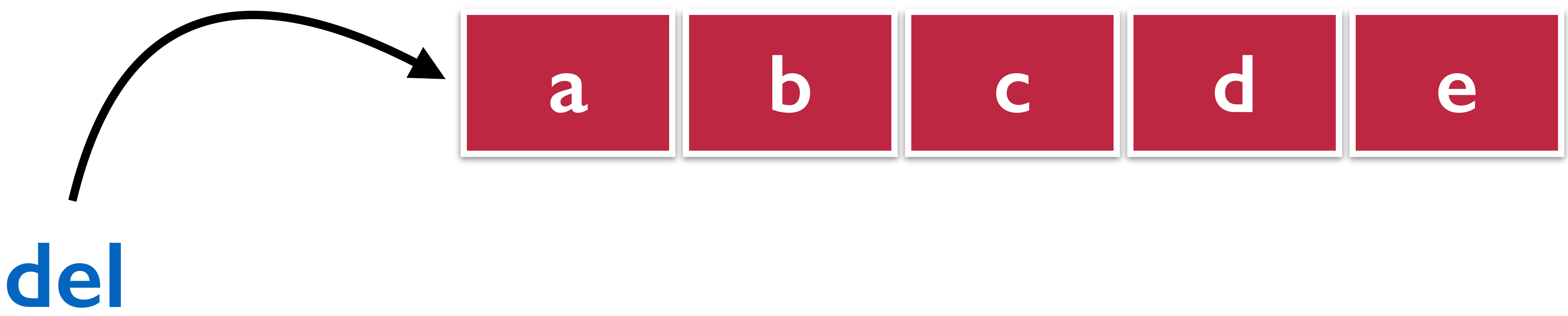

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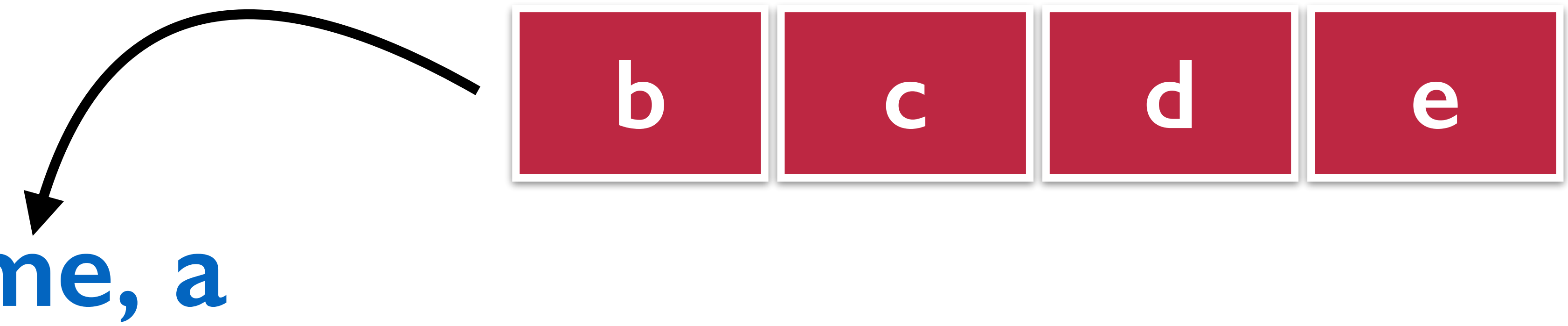

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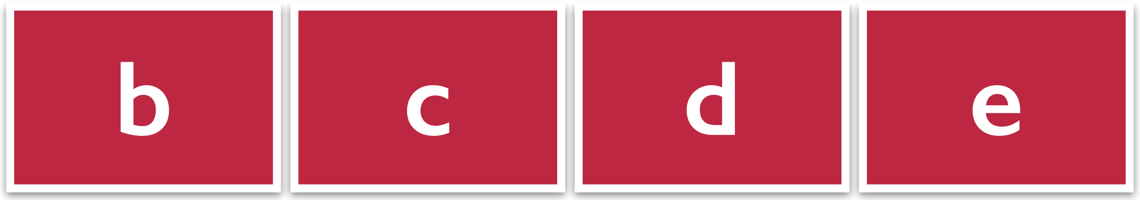

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When are the none and some branches taken?
Can the size of queue be encoded in the type?

Work done by Queue

Count the total number of messages!

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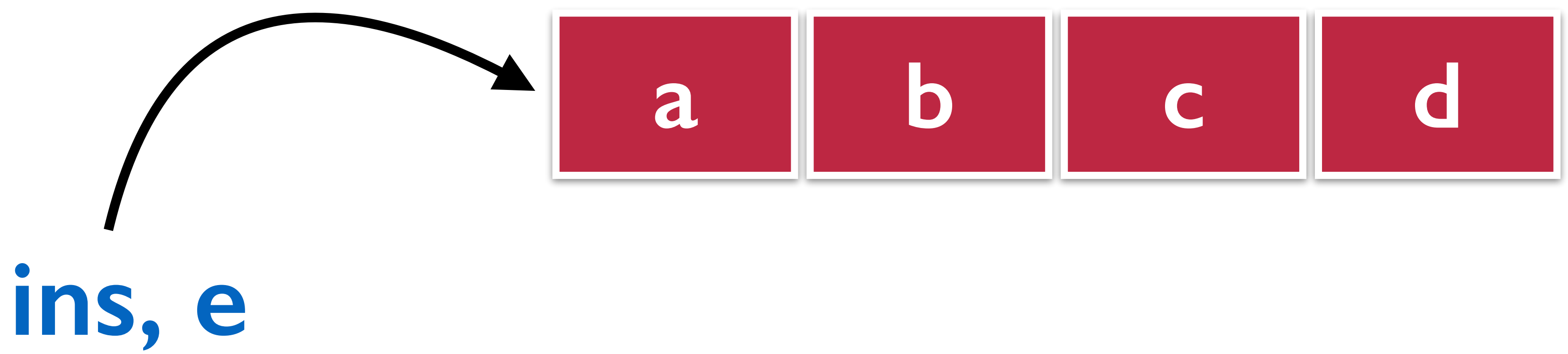

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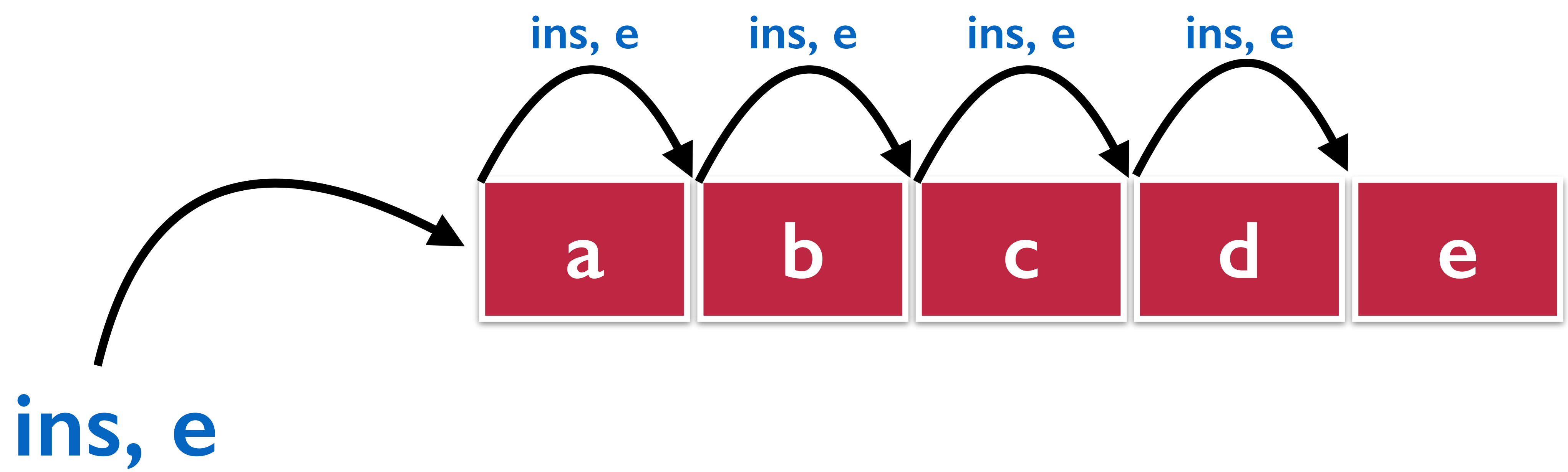

w_i = Work done to process insertion
= 2n (n is the size of queue)
= 'ins' and 'e' travel to end of queue

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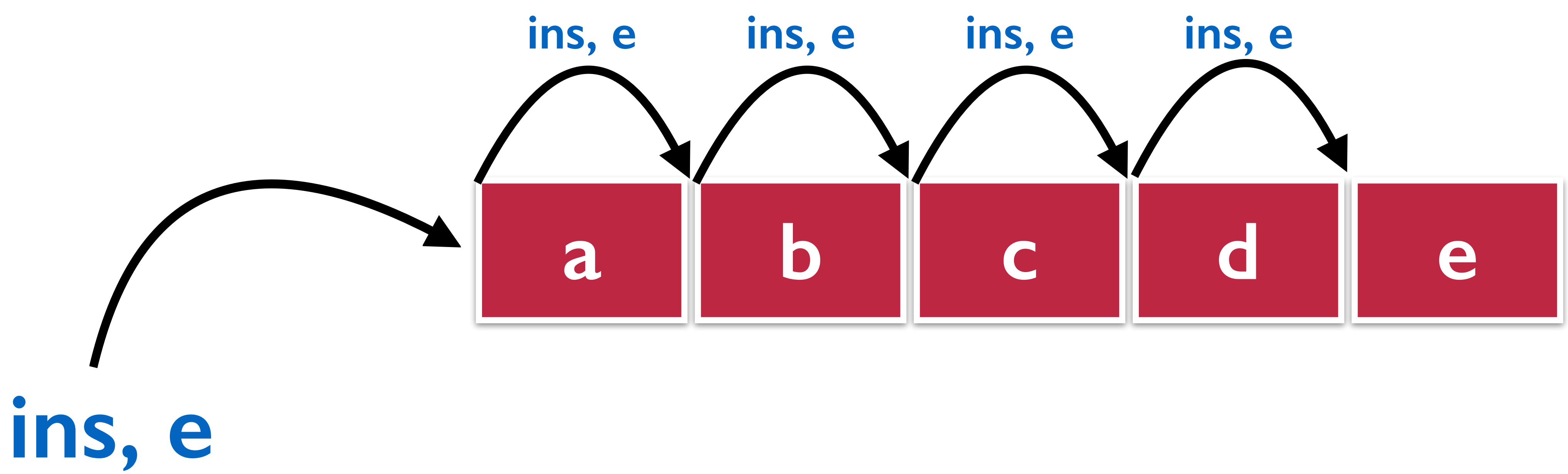

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Insertion: How do you refer to n in the queue type?

Refined Queue Type

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\begin{aligned}
\mathrm{queue}_A[n] = \&\{\mathbf{ins} : A \multimap \mathrm{queue}_A[n+1],\\
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Index Refinement
(Size of Queue)

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Index Refinement (Size of Queue)

Proof Constraints (Sent by queue)

- ‘none’ branch: send (proof of) constraint $\{n=0\}$
- ‘some’ branch: send (proof of) constraint $\{n>0\}$
- Domain of constraints: Presburger Arithmetic

Three Key Results

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Typing System with Arithmetic Refinements

Negative Result:
Type equality with refinements is undecidable (even though type equality for simple session types and Presburger arithmetic are both decidable!)

Positive Result:
A sound algorithm for type equality (works exceptionally well in practice!)

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$?\{\phi\}.\ A$

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Client receives (proof of) constraint ϕ	Client sends (proof of) constraint ϕ

Since Presburger arithmetic is decidable and only closed programs are executed, no need to exchange proofs/constraints at runtime

Type Grammar

$$
\begin{array}{rcl}
A & ::= & \oplus\{\ell : A\}_{\ell \in L} \mid \&\{\ell : A\}_{\ell \in L} \mid A \otimes A \mid A \multimap A \mid \mathbf{1} \\
& \mid & V[\overline{e}] \mid ?\{\phi\}.\, A \mid !\{\phi\}.\, A \mid \exists n.\, A \mid \forall n.\, A
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Type Grammar

Standard session types

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Send / Recv proof constraints

Send / Recv natural numbers

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Co-inductive Type Equality

- Two types are considered equal if they have the *same communication behavior (structural type system)*
- Formally defined by establishing a *type bisimulation*
- Refinement session types can encode *2-counter machines (2CM)*!
- Equality reduces to determining if the 2CM halts, and is thus, *undecidable!*
- Given a 2CM and input, we construct two types that *only differ at the halting instruction*

Reduction from 2CM

Increment Instruction:
m : inc(c_1) ; goto k

$$A_m[c_1, c_2] = \oplus\{\mathbf{inc}_1 : A_k[c_1 + 1, c_2]\}$$

$$A'_m[c_1, c_2] = \oplus\{\mathbf{inc}_1 : A'_k[c_1 + 1, c_2]\}$$

Reduction from 2CM

Decrement Instruction:
m : zero(c₁)? goto k : goto n

$$A_m[c_1, c_2] = \oplus\{\mathbf{zero_1} : ?\{c_1 = 0\}.\, A_k[c_1, c_2], \\ \mathbf{dec_1} : ?\{c_1 > 0\}.\, A_n[c_1 - 1, c_2]\}$$

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Reduction from 2CM

Halting Instruction:
m : halt

$$A_m[c_1, c_2] = A$$

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The types are equal iff the 2CM does not halt

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Equality Algorithm

$$\mathrm{ctr}[x, y] = \oplus\{\mathbf{lt} : ?\{x < y\}.\, \mathrm{ctr}[x+1, y],\\ \mathbf{ge} : ?\{x \geq y\}.\, \mathbf{1}\}$$

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Goal: Find a counterexample

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store the equality constraint and expand both types

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lt branch

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substitute *x+1* for *X*, *y* for *Y*

$$\dfrac{\dfrac{x < y \equiv x+1 < y+1 \qquad \mathrm{ctr}[x+1,y] \equiv \mathrm{ctr}[x+2,y+1]}{?\{x<y\}.\,\mathrm{ctr}[x+1,y] \equiv ?\{x+1<y+1\}.\,\mathrm{ctr}[x+2,y+1]}}{\mathrm{ctr}[x,y] \equiv \mathrm{ctr}[x+1,y+1]}$$

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Formal Judgment

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Free variables

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Constraint satisfied by V

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Types A and B are equal under constraint C

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Constraint satisfied by V

Equality constraints stored

$$\mathcal{V} \; ; \; C \; ; \; \Gamma \vdash A \equiv B$$

Types A and B are equal under constraint C

$$x, y \; ; \; \top \vdash \mathrm{ctr}[x, y] \equiv \mathrm{ctr}[x + 1, y + 1]$$

Closing the Loop

$$\frac{\langle \mathcal{V}' \; ; \; C' \; ; \; V_1[\overline{E_1}] \equiv V_2[\overline{E_2}] \rangle \in \Gamma \qquad \forall \mathcal{V} . C \Rightarrow \exists \mathcal{V}' . C' \wedge \overline{E_1} = \overline{e_1} \wedge \overline{E_2} = \overline{e_2}}{\mathcal{V} \; ; \; C \; ; \; \Gamma \vdash V_1[\overline{e_1}] \equiv V_2[\overline{e_2}]} \; \mathsf{def}$$

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Closing the Loop

Already encountered constraint

$$\frac{\langle \mathcal{V}' \; ; \; C' \; ; \; V_1[\overline{E_1}] \equiv V_2[\overline{E_2}] \rangle \in \Gamma \qquad \forall \mathcal{V} . C \Rightarrow \exists \mathcal{V}' . C' \land \overline{E_1} = \overline{e_1} \land \overline{E_2} = \overline{e_2}}{\mathcal{V} \; ; \; C \; ; \; \Gamma \vdash V_1[\overline{e_1}] \equiv V_2[\overline{e_2}]} \; \text{def}$$

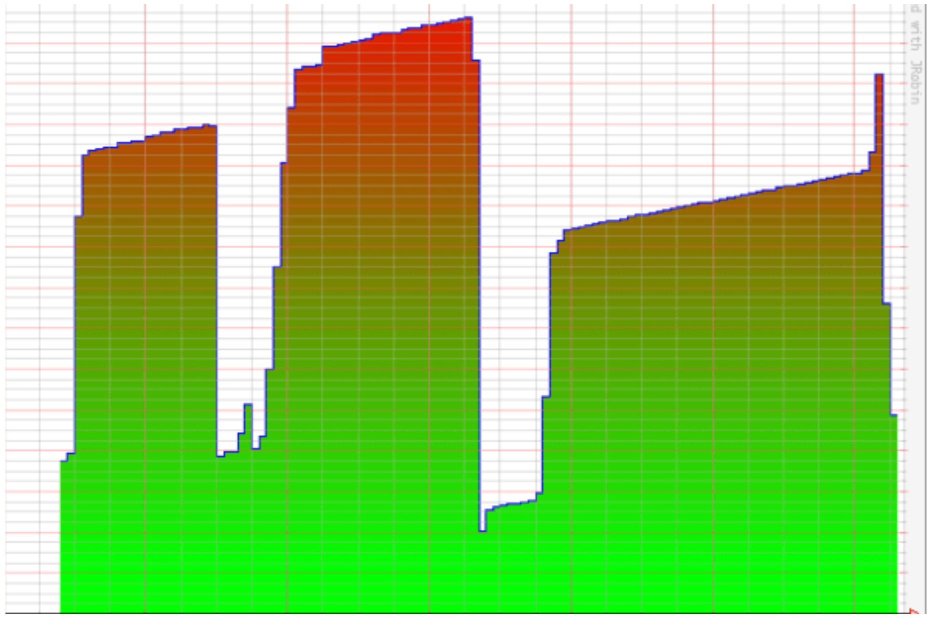
If we know $V_1[E_1] \equiv V_1[E_2]$,
can we prove $V_1[e_1] \equiv V_2[e_2]$?
That is what the second premise achieves!

Rast Programming Language

Goal of Rast

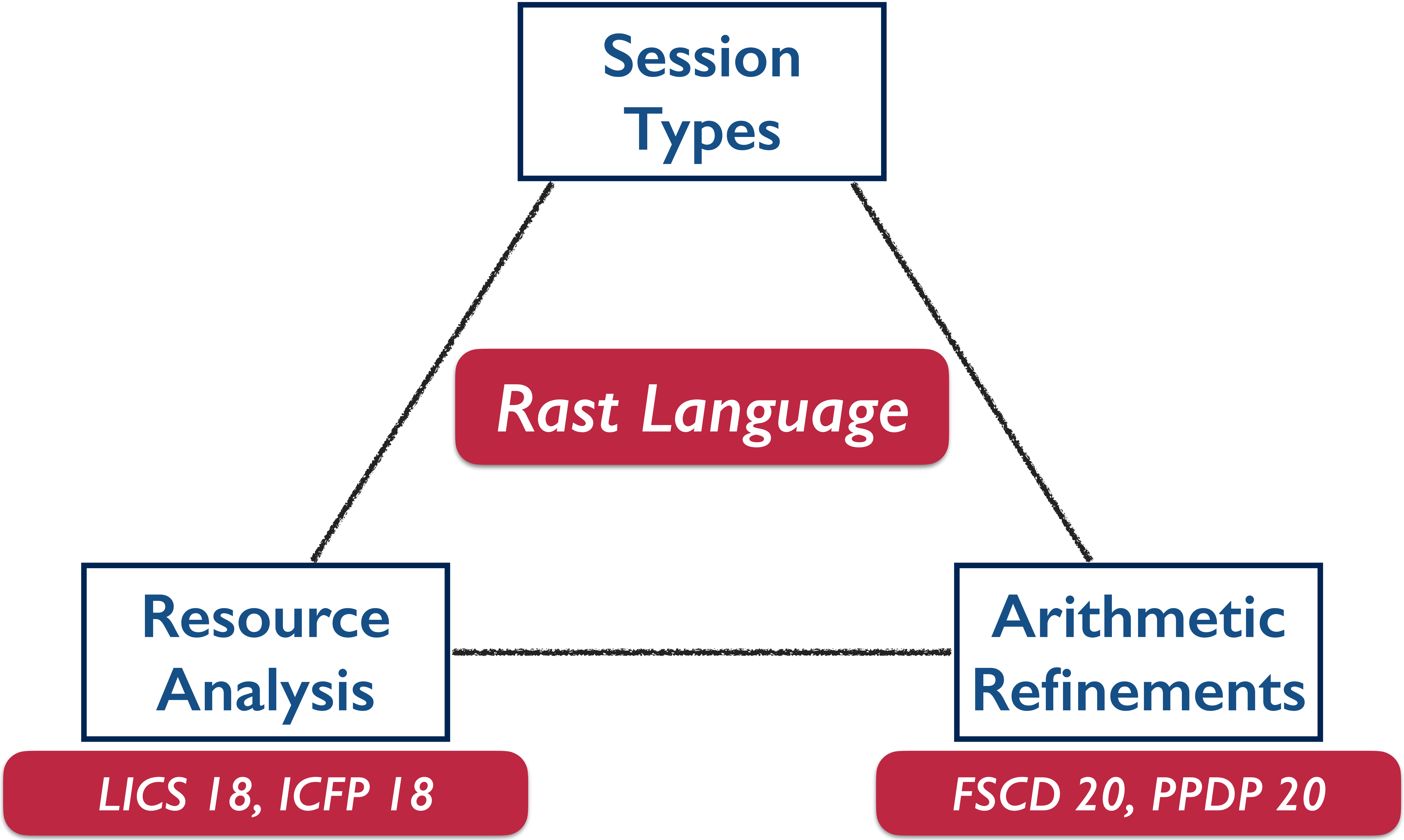
Lightweight Verification and Resource Analysis of Concurrent Programs

Execution Time



Memory Usage

Key Features of Rast



Evaluation

Module	LOC	#Defs	T (ms)
arithmetic	143	8	1.325
integers	114	8	1.074
linlam	67	6	4.003
list	441	29	3.419
primes	118	8	1.646
segments	65	9	0.195
ternary	235	16	1.967
theorems	141	16	0.894
tries	308	9	5.283
Total	**1632**	**109**	**19.806**

Conclusion

Conclusion

Typing System with Arithmetic Refinements

Negative Result:
Type equality with refinements is undecidable

Positive Result:
A sound algorithm for type equality

Meta Result:
Results generalize to any structural type system e.g. functional programming languages