CS 599 A1: Assignment 6

Due: Thursday, April 30, 2026

Total: 100 pts

Instructor: Ankush Das

- This assignment is due midnight on the above date and must be submitted electronically on Gradescope.
- You are provided a tex file, named asgn6.tex. It contains environments called solution to enter your solutions. And please include your name and BU ID in the author section (above).
- Although not recommended, you can submit handwritten answers scanned as a PDF and clearly legible.

1 Commutativity and Associativity

A binary operator $*$ is defined to be commutative if $A * B$ and $B * A$ are equivalent. In other words, $A * B \Longrightarrow B * A$ and $B * A \Longrightarrow A * B$.
A binary operator $*$ is defined to be associative if $A * (B * C)$ and $(A * B) * C$ are equivalent. In other words, $A * (B * C) \Longrightarrow (A * B) * C$ and $(A * B) * C \Longrightarrow A * (B * C)$.

Problem 1 (Mandatory; No LLMs; 10 pts) *Refer the rules in Appendix A. Which operators in ordered logic are commutative? For each operator $*$ that is commutative, provide the proofs of equivalence of $A * B$ and $B * A$, i.e., $A * B \Longrightarrow B * A$ and $B * A \Longrightarrow A * B$.*

Problem 2 (Mandatory; No LLMs; 20 pts) *Refer the rules in Appendix A. Which operators in ordered logic are associative? For each operator $*$ that is associative, provide the proofs of equivalence of $A * (B * C)$ and $(A * B) * C$, i.e., $A * (B * C) \Longrightarrow (A * B) * C$ and $(A * B) * C \Longrightarrow A * (B * C)$.*

2 Invertibility

In this problem, you need to determine which inference rules in ordered logic are invertible. Recall the definition of an invertible inference rule: if the conclusion of the rule is provable, then all the premises must be provable. In other words, given a derivation of the conclusion of a rule, you can construct a derivation of each premise of the rule.

Problem 3 (Mandatory; No LLMs; 40 pts) *Which inference rules in Appendix A are invertible? For each rule that is invertible, provide a proof of its invertibility.*
Hint: *Use the cut admissibility theorem.*

3 New Operators

Problem 4 (Mandatory; No LLMs; 10 pts) *Suppose we change the $\backslash L$ to the following:*

$$\frac{\Omega_L \; B \; \Omega_R \Longrightarrow C}{\Omega_L \; A \; (A\backslash B) \; \Omega_R \Longrightarrow C} \; \backslash L^*$$

Instead of allowing an arbitrary Ω to the left of $(A\backslash B)$, we require that the proposition on the left of $(A\backslash B)$ must be A. Based on this, answer the following questions:

1. *Does this violate identity expansion? If yes, provide a brief justification. If not, provide a derivation of identity expansion.*
2. *Does this violate cut admissibility? If yes, provide a brief justification. If not, provide a derivation of cut admissibility.*
3. *Give an example of a proposition that is derivable in ordered logic but no longer derivable if we use $\backslash L^*$ instead of $\backslash L$.*

Problem 5 (Mandatory; No LLMs; 10 pts) *We define a new operator $A \diamond B$ with the following left and right rules:*

$$\frac{\Omega_L \Longrightarrow A \qquad \Omega_R \Longrightarrow B}{\Omega_L \; \Omega \; \Omega_R \Longrightarrow A \diamond B} \diamond R \qquad\qquad \frac{\Omega_L \; A \; \Omega \; B \; \Omega_R \Longrightarrow C}{\Omega_L \; (A \diamond B) \; \Omega_R \Longrightarrow C} \diamond L$$

Does this operator satisfy identity expansion? Does it satisfy cut admissibility? In each case, provide a proof if the property holds, and a brief justification if it does not.

Problem 6 (Mandatory; No LLMs; 10 pts) *We define a new operator $A \Diamond B \Box C$ with the following right rule:*

$$\frac{\Omega_1 \; \Omega_2 \Longrightarrow A \qquad \Omega_1 \Longrightarrow B \qquad \Omega_2 \Longrightarrow C}{\Omega_1 \; \Omega_2 \Longrightarrow A \Diamond B \Box C} \Diamond\Box R$$

Design appropriate left rule(s) for $A \Diamond B \Box C$ such that identity expansion and cut admissibility hold for this operator. You do not need to show the proofs of these properties. If it is not possible to design correct left rule(s), please provide a brief justification.

4 Optional Problems

Problem 7 (Optional; No LLMs; 20 pts) *Prove identity expansion (Theorem 1) in ordered logic.*

Problem 8 (Optional; No LLMs; 30 pts) *Provide a (partial) proof of cut admissibility (Theorem 2) in ordered logic. You only need to show all the principal and commutative cases of / operator (over).*

A Ordered Logic Inference Rules

Fuse ($\bullet$)

$$\frac{\Omega_L \Longrightarrow A \qquad \Omega_R \Longrightarrow B}{\Omega_L\ \Omega_R \Longrightarrow A \bullet B}\ \bullet R \qquad\qquad \frac{\Omega_L\ A\ B\ \Omega_R \Longrightarrow C}{\Omega_L\ (A \bullet B)\ \Omega_R \Longrightarrow C}\ \bullet L$$

Under ($\backslash$)

$$\frac{A\ \Omega \Longrightarrow B}{\Omega \Longrightarrow A\backslash B}\ \backslash R \qquad\qquad \frac{\Omega \Longrightarrow A \qquad \Omega_L\ B\ \Omega_R \Longrightarrow C}{\Omega_L\ \Omega\ (A\backslash B)\ \Omega_R \Longrightarrow C}\ \backslash L$$

Over ($/$)

$$\frac{\Omega\ A \Longrightarrow B}{\Omega \Longrightarrow B/A}\ /R \qquad\qquad \frac{\Omega \Longrightarrow A \qquad \Omega_L\ B\ \Omega_R \Longrightarrow C}{\Omega_L\ (B/A)\ \Omega\ \Omega_R \Longrightarrow C}\ /L$$

With ($\&$)

$$\frac{\Omega \Longrightarrow A \qquad \Omega \Longrightarrow B}{\Omega \Longrightarrow A \,\&\, B}\ \&R \qquad \frac{\Omega_L\ A\ \Omega_R \Longrightarrow C}{\Omega_L\ (A \,\&\, B)\ \Omega_R \Longrightarrow C}\ \&L_1 \qquad \frac{\Omega_L\ B\ \Omega_R \Longrightarrow C}{\Omega_L\ (A \,\&\, B)\ \Omega_R \Longrightarrow C}\ \&L_2$$

Plus ($\oplus$)

$$\frac{\Omega \Longrightarrow A}{\Omega \Longrightarrow A \oplus B}\ \oplus R_1 \qquad \frac{\Omega \Longrightarrow B}{\Omega \Longrightarrow A \oplus B}\ \oplus R_2 \qquad \frac{\Omega_L\ A\ \Omega_R \Longrightarrow C \qquad \Omega_L\ B\ \Omega_R \Longrightarrow C}{\Omega_L\ (A \oplus B)\ \Omega_R \Longrightarrow C}\ \oplus L$$

Unit (1)

$$\frac{}{\cdot \Longrightarrow 1}\ 1R \qquad\qquad \frac{\Omega_L\ \Omega_R \Longrightarrow C}{\Omega_L\ 1\ \Omega_R \Longrightarrow C}\ 1L$$

Identity

$$\frac{}{p \Longrightarrow p}\ \textsc{Id}$$

Theorem 1 (Identity Expansion) *For an arbitrary proposition A, $A \Longrightarrow A$ is derivable.*

Theorem 2 (Cut Admissibility) *Given derivations of $\Omega \Longrightarrow A$ and $\Omega_L\ A\ \Omega_R \Longrightarrow C$, we can construct a derivation for $\Omega_L\ \Omega\ \Omega_R \Longrightarrow C$.*